Composition Relation between Gap Solitons and Bloch Waves in Nonlinear Periodic Systems

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We show with numerical computation and analysis that Bloch waves, at either the center or edge of the Brillouin zone, of a one dimensional nonlinear periodic system can be regarded as infinite chains composed of fundamental gap solitons (FGSs). This composition relation between Bloch waves and FGSs leads us to predict that there are n families of FGSs in the nth band gap of the corresponding linear periodic system, which is confirmed numerically. Furthermore, this composition relation can be extended to construct a class of solutions similar to Bloch waves but with multiple periods.

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With the advance of experimental techniques, a wide variety of nonlinear periodic systems has arisen in recent years. They include nonlinear waveguide arrays [1,2], optically induced photonic lattices [3,4], and Bose-Einstein condensates (BECs) in optical lattices [5,6]. These systems have been intensively researched, ranging from superfluidity [7] and instability [7–9] to the generation of solitons [1–6] and vortex [10].

In these nonlinear periodic systems, there are two completely different kinds of stationary states: Bloch wave and gap solitons. Bloch waves, which exist in both linear and nonlinear periodic systems, are extensive solutions that spread over the whole system [7]. In contrast, gap solitons exist only in nonlinear periodic systems and are localized in space [3]. In this work, we focus primarily on a class of solitons called fundamental gap solitons (FGSs), whose main peaks are localized inside one unit cell [11].

We show that a relation, which we call composition relation, exists between Bloch waves and FGSs. In this relation, the FGSs can be viewed as building blocks for Bloch waves at either the center or the edge of the Brillouin zone (BZ) that have the same propagation constants (or chemical potentials) as FGSs. With this composition relation between Bloch waves and gap solitons, we predict that there are n families of FGSs in the nth band gap of the corresponding linear periodic system. This is confirmed by our numerical computation. Only two families of FGSs were known [11,12]: one is called fundamental gap soliton and the other subfundamental gap soliton [13]. From how these two different FGSs are named, it seems that people have not been expecting other types of FGSs. Furthermore, we can generalize this composition relation to construct with FGSs extensive stationary states, which are similar to Bloch waves but with multiple periods.

Consider a one dimensional defocusing nonlinear periodic system described by the equation,

$$i\frac{\partial\Psi}{\partial\xi} = -\frac{1}{2}\frac{\partial^2\Psi}{\partial x^2} + V(x)\Psi + |\Psi|^2\Psi, \quad (1)$$

where $V(x)$ is a periodic potential. When ξ is z, the above equation describes a light wave propagating along the z direction in a nonlinear media that is periodic along the x direction [1,2]. If ξ is time t, then the equation describes a BEC in an optical lattice [6,7]. Without loss of generality, we choose $\xi = z$ and $V(x) = \nu\cos(x)$.

Both Bloch waves and gap solitons are stationary solutions of Eq. (1), which are of the form $\Psi(x,\xi) = \phi(x)\times\exp(-i\mu\xi)$. μ is called propagation constant for light wave (or chemical potential for a BEC). $\phi(x)$ obeys the z-independent equation,

$$-\frac{1}{2}\frac{\partial^2\phi}{\partial x^2} + V(x)\phi + |\phi|^2\phi = \mu\phi. \quad (2)$$

Without the cubic term, the above equation is a linear periodic system and admits only Bloch wave solutions. Bloch waves are defined as $\phi(x) = \exp(ikx)\psi_{n,k}(x)$ with $\psi_{n,k}(x) = \psi_{n,k}(x+2\pi)$, where k is the Bloch wave vector and n is the band index. Associated with each Bloch wave $\psi_{n,k}(x)$, there is a propagation constant $\mu_n(k)$. As k varies through the BZ, $\mu_n(k)$ makes up Bloch bands. In Fig. 1(a), we have plotted the two lowest Bloch bands LB_1 and LB_2 for the linear periodic system. Notice that there is a large

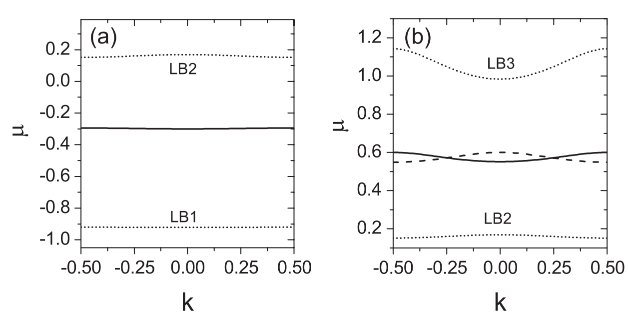

FIG. 1. Linear and nonlinear Bloch bands for $\nu = 1.5$. Dotted lines are linear Bloch bands. Label LBi stands for the ith linear Bloch band. (a) The first nonlinear band (solid line) in the first linear band gap with $\mathcal{N} = 1.6908$; (b) The second (dashed line) and first (solid line) nonlinear bands in the second band gap with $\mathcal{N} = 1.5268$ and 4.8437, respectively.

band gap between the two linear Bloch bands, where no physical solution is allowed.

The nonlinear term in Eq. (2) does not destroy these Bloch waves and Bloch bands but only modifies them. This nonlinear effect is completely determined by the norm $\mathcal{N} = \int_0^{2\pi} |\psi_k(x)|^2 dx$, which can be regarded as the strength of nonlinearity. In Fig. 1(a), the lowest nonlinear Bloch band $\mu_1(k)$ is plotted for $\mathcal{N} = 1.6908$. Clearly, the Bloch band has moved up relative to its linear counterpart LB_1 due to nonlinearity, and its shape has also changed. If $\mathcal{N}$ is lowered to zero, this band $\mu_1(k)$ will move down and be reduced to LB_1. When $\mathcal{N}$ is increased, $\mu_1(k)$ will move up continuously with no limit. Nonlinear Bloch waves and Bloch bands are found numerically with the method in Ref. [8].

Besides Bloch waves, Eq. (2) has also gap soliton solution. However, for a gap soliton, its μ can only take values inside the linear band gaps. This is why it is called gap soliton and why it only exists in nonlinear periodic systems. Gap soliton solutions are found by integrating Eq. (2) numerically using the Newton relaxation method [13,14]. To check the convergence of the results, we substitute the found gap soliton solutions into Eq. (2) and compute the maximum difference between the left and right sides of Eq. (2).

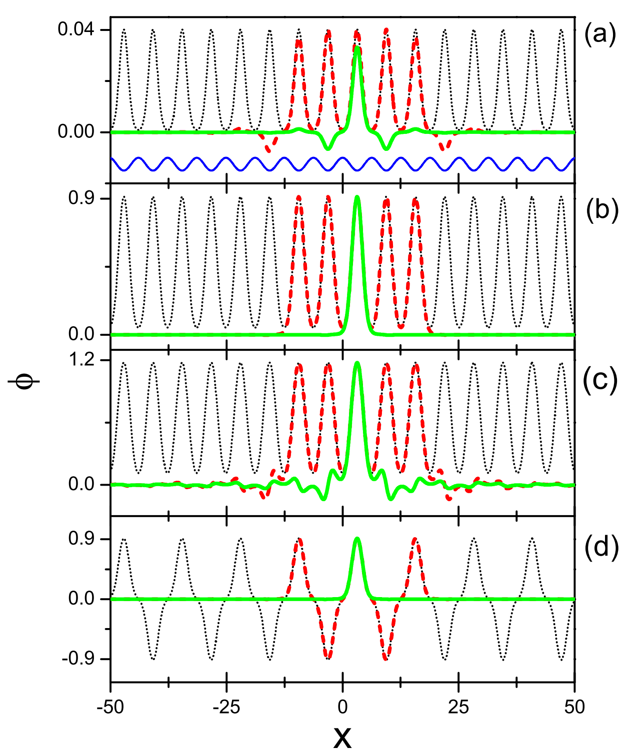

FIG. 2 (color online). Bloch waves of the first nonlinear band and FGSs in the first band gap. Dotted (black), solid (green), and dashed (red) lines represent Bloch wave, FGS, and gap wave, respectively. $\nu = 1.5$. The Bloch waves in (a), (b), and (c) are at the BZ center and the Bloch wave in (d) is at the BZ edge. (a) $\mathcal{N} = 0.0027$, $\mu = -0.92$, line at bottom indicates potential; (b) $\mathcal{N} = 1.6908$, $\mu = -0.3$; (c) $\mathcal{N} = 3.2194$, $\mu = 0.151$; (d): $\mathcal{N} = 1.6738$, $\mu = -0.3$.

We discover that the localized FGS and the extensive Bloch wave are related to each other. We consider first the situation in Fig. 1(a), where the first nonlinear Bloch band lies entirely inside the first linear band gap. We focus on the Bloch wave at the center of the BZ. Since its propagation constant μ lies in the band gap, there exists a FGS for this value of μ. In Fig. 2(b), we have plotted the Bloch wave and the corresponding FGS together. We notice that the FGS matches very well with the individual peak of the Bloch wave. This direct observation suggests that a Bloch wave at the center of the BZ can be viewed as a chain of FGSs pieced together. We have also compared FGSs and Bloch waves for other parameters in Figs. 2(a) and 2(c). Figure 2(a) is for the case of μ near the top of the first linear Bloch band LB_1; Fig. 2(c) is for the case of μ close to the bottom of the second linear Bloch band LB_2. It is apparent that the matching between the FGS and the Bloch wave in Fig. 2(a) is not as good as in the other two cases. This mismatch implies that there is no simple mathematical expression for this relation. Later, we shall show that this relation can lead to predictions that are confirmed by our numerical computation.

A similar relation exists between a Bloch wave at the edge of the BZ and a FGS. The only difference is that the FGS is put together with alternative signs as shown in Fig. 2(d). For ease of reference, we call this relation composition relation. In fact, a finite number of FGSs can also be used to form a bigger gap soliton as shown in Fig. 2. These bigger gap solitons are called gap waves in Ref. [15], which can be regarded as intermediate states between FGSs and Bloch waves.

We turn to the second linear band gap, which presents a key difference: two different nonlinear Bloch bands can be lifted into this gap. As one increases continuously the norm $\mathcal{N}$ from zero, first the second nonlinear Bloch band $\mu_2(k)$ will be moved up into the second band gap; then the first nonlinear Bloch band $\mu_1(k)$ will also be lifted into the second band gap as shown in Fig. 1(b) when $\mathcal{N}$ is large enough. Since the Bloch waves in these two different bands are very distinct characteristically, one needs two different families of FGSs to construct them if the composition relation between FGS and Bloch waves also holds in the second linear band gap. Two different FGSs indeed exist as reported in Refs. [11,12]. In Fig. 3, we have plotted the Bloch waves of the second nonlinear band and the corresponding FGS, which was called subfundamental gap soliton in Ref. [13]. The matching is very good as in the first band gap.

Expecting this trend to hold for any band gap, we predict that there are n different families of FGSs inside the nth linear band gap. As the norm $\mathcal{N}$ increases, all the nonlinear Bloch bands $\mu_m(k)$ with $m \leq n$ will move successively into the nth band gap. To construct the Bloch waves

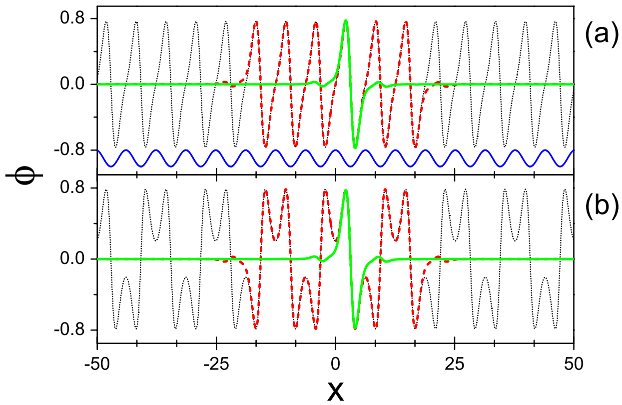

FIG. 3 (color online). Bloch waves of the second nonlinear band and FGSs of the second family in the second band gap. $\nu = 1.5$, $\mu = 0.6$. Dotted (black), solid (green), and dashed (red) lines represent Bloch wave, FGS, and gap wave, respectively. (a) Bloch wave at the center of the BZ, $\mathcal{N} = 1.5268$. Line at bottom mimics potential. (b) Bloch wave at the edge of the BZ, $\mathcal{N} = 1.7417$.

belonging to these n different bands, we need n different families of FGSs. This prediction is confirmed by our numerical computation for the 3rd band gap. In Fig. 4, the third FGS is shown with the corresponding Bloch waves. To our best knowledge, this third family of FGSs has not been reported before. In order to observe experimentally gap solitons, the initial input pattern should be close to desired soliton profiles [3]. So, to observe the third family FGS, one can use two localized laser beams, whose wavelength is much shorter than the period of the waveguide, to form an interference pattern with three large peaks in a unit cell of the periodic waveguide.

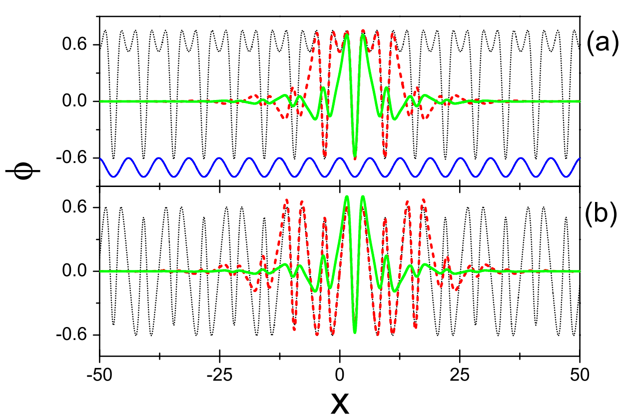

FIG. 4 (color online). Bloch waves of the third nonlinear band and FGSs of the third family in the third band gap. $\nu = 1.5$, $\mu = 1.4$. Dotted (black), solid (green), and dashed (red) lines represent Bloch wave, FGS, and gap wave, respectively. (a) Bloch wave at the center of the BZ, $\mathcal{N} = 2.0521$. Line at bottom mimics potential. (b) Bloch wave at the edge of the BZ, $\mathcal{N} = 1.0242$.

One may have noticed that the first family of FGSs has one peak in a unit cell, the second family has two peaks, and the third has three. This can be explained as follows. The Bloch wave in the first Bloch band, which uses the first family of FGSs as its building block, comes from the ground state of an individual well of the periodic potential $V(x)$. The Bloch wave of the second band, which is constructed with the second family of FGSs, originates from the first excited state of an individual well; and so on. As the nth bound state has $(n - 1)$ nodes, the nth family of FGSs should have n peaks. This analysis also indicates that the composition relation is similar to the relation between Bloch waves and bound states of an individual well in linear periodic systems.

This composition relation has another prediction. From the band structure shown in Fig. 1, we see that the norm $\mathcal{N}$ has to be over a critical value to lift the first nonlinear Bloch band into the second band gap while there is no such critical value for moving the second nonlinear band into the second band gap. This is true in general: one has to increase $\mathcal{N}$ over a critical value to move the mth nonlinear Bloch band into the nth $(n > m)$ linear band gap while there is no critical value to move the nth nonlinear Bloch band into the nth gap. This observation, combined with the composition relation between Bloch wave and FGS, leads us to predict that the norm of the mth family of FGSs in the nth band gap is always bigger than a critical value. For a gap soliton, its norm N is defined as $N = \int_{-\infty}^{\infty} |\phi(x)|^2 dx$. We have computed numerically how N of FGSs changes with μ and plotted it in Fig. 5, where the prediction is seen confirmed. For example, the second family of FGSs in the third band gap has always $N > 5.0152$. We have also plotted how the norm $\mathcal{N}$ of Bloch waves changes with μ. As seen in Fig. 5, N and $\mathcal{N}$ are very close to each other and have similar dependence on μ. Note that the mth

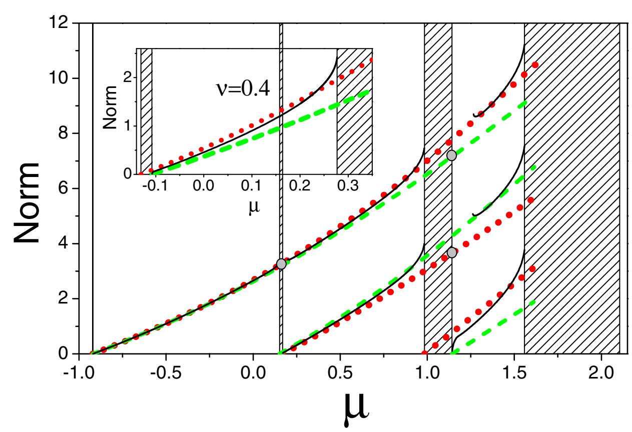

FIG. 5 (color online). Norms of both FGSs and nonlinear Bloch waves at the center and edge of the BZ as a function of μ. $\nu = 1.5$. Shaded areas are linear bands. Dotted (red), dashed (green), solid (black) lines represent Bloch waves at the BZ center, Bloch waves at the BZ edge, and FGSs, respectively. Inset figure is for the first band gap at a lower potential $\nu = 0.4$.

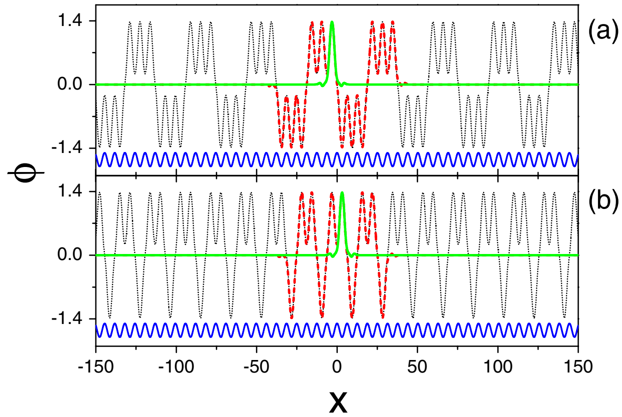

FIG. 6 (color online). Two types of triple-periodic solutions. Dotted (black), dashed (red), and solid (green) lines are for triple-periodic solution, bigger soliton composed by FGS finitely, and FGS, respectively. Lines at the bottom of each figure mimic potential. $\nu = 1.5$, $\mu = 0.6$.

family of FGSs in the nth gap does not exist for all values of μ inside the gap. The existence of such a cutoff on μ for gap solitons has been reported before in Refs. [12,15].

The intuitive picture that larger gap solitons, such as gap waves [15], can be viewed as a chain of FGSs has been floating in the community [11,14]. However, since there is no simple mathematical expression for this relation, there is always some doubt that this relation truly exists. What we have done is to firmly establish this kind of relation between Bloch waves and FGSs with *analysis* and numerical computation. It is interesting to note that there exist the periodic solutions for a nonlinear Schrödinger equation with no external potential and these solutions can be viewed as trains of dark or gray solitons as well [16].

The stabilities of gap solitons and gap waves are analyzed. Our analysis indicates that the first family FGSs shown in Fig. 5 are mostly stable. They become unstable only in a small area near the band edges in the third gap. The stabilities of the second family of FGSs in the second gap and the third family of FGSs in the third gap are similar: both of them are stable in the areas with smaller propagation constant and are unstable beyond a critical value of μ. All the second family of FGSs in the third gap are unstable. The stability properties of gap waves are closely related to their corresponding Bloch waves and are independent of how many wells they occupy. Our stability analysis is confirmed by the beam propagation method with initial random noise.

This composition relation between gap soliton and Bloch wave can be generalized to construct new solutions. In constructing Bloch waves from the FGS, FGSs are arranged with either the same or alternative signs between neighboring pairs. We find that other arrangements of signs between neighboring pairs of FGSs can lead to multiperiodic solutions for Eq. (1). For example, we are able to recover all the even-periodic solutions found in Ref. [17]. We are also be able to find the solutions of odd periods, which were speculated to exist in Ref. [17]. Two solutions of triple periods are shown in Fig. 6. Unfortunately, both of the solutions are unstable.

To confirm the relation existing in any periodic media, we have repeated our study for a periodic potential $V(x)$ that describes waveguide arrays profile [1,2]. The results are essentially the same, indicating that the detailed profile of $V(x)$ is not important.

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