



Time-boundary scattering and topological resonant transmissions

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Time boundaries (TBs), temporal analogues of spatial interfaces, offer a powerful handle to engineer quantum systems. However, unlike the well-developed stationary scattering theory at spatial interfaces, a unified framework for quantum scattering at TBs has been missing. Here we develop a Bloch-wave scattering theory for TBs by introducing a temporal scattering matrix S between incoming and outgoing Bloch channels. We uncover topological resonant transmissions (RTs)—poles of S that yield perfect interband transmission and dynamical freezing of the quantum state. We establish a bulk–time–boundary correspondence for all integer Altland–Zirnbauer classes: the number of RTs equals the jump of the bulk topological invariant across the TB. In one dimension this gives a time-domain Levinson’s theorem. A topological analysis further reveals a striking dimensional dependence. In even dimensions RTs are robust to temporal modulations and disorder, whereas in odd dimensions they can be destroyed by dynamical symmetry breaking. Our work places temporal and spatial scattering on the same footing and opens avenues for engineering and probing quantum dynamics.

time-boundary | topological phase | resonant transmission | scattering matrix

Classical waves crossing a spatial interface undergo reflection and refraction, as dictated by Fermat’s principle. A time boundary (TB) is the temporal analogue of a spatial interface, realized for example in optics as an abrupt change of the refractive index in time. The space–time duality of Maxwell’s equations implies that waves encountering TBs experience time reflection and refraction (1–6). Unlike spatial boundaries which conserve energy but alter momentum, a TB conserves momentum while changing energy due to broken time-translation symmetry. Classical TB effects have been observed in microwave and optical systems (7–11), acoustics (12), and water waves (13), with applications in broadband frequency translation (9, 10), nonreciprocal devices (14), and photonic time crystals (15–19).

In the quantum realm, wave propagation obeys the Schrödinger equation, $i\partial_t|\Psi(t)\rangle = H|\Psi(t)\rangle$. Reflection and transmission at a spatial interface are governed by stationary scattering theory. The simplest TB is a sudden quench of the Hamiltonian in time, which generates a wealth of nonequilibrium phenomena and is widely used as a tool for quantum control. Extensive work on quench dynamics has revealed rich spatiotemporal topological patterns (20–42) for topological band systems (43, 44). In most of these studies the focus is on the evolution of quantum states and dynamical observables. The quench is treated primarily as a “black-box” drive for quantum dynamics, rather than as a TB with its own scattering mechanism. Very recently, quench-induced time reflection/refraction was observed in a one-dimensional (1D) Su–Schrieffer–Heeger lattice of cold atoms (45). Yet quantum scattering at TBs remains far less understood vis-à-vis its well-established spatial counterpart. Unlike classical waves, quantum spatial and temporal scattering breaks space–time duality, since the Schrödinger equation is second order in space but only first order in time. Moreover, core scattering notions at TBs, including a well-defined scattering matrix, its poles, and their relation to spatial scattering and topology, have remained elusive.

From a practical standpoint, most previous studies of TBs in classical and quantum systems have focused on sharp, abrupt parameter changes, which are experimentally demanding and restrictive. In reality, TBs can be much more flexible, generic, and experimentally friendly. For instance, they can have finite duration, or follow smooth parameter ramps (46, 47) or periodic modulations as in photonic time crystals (15–19). Such versatile forms of temporal engineering are now routine in platforms like cold atoms (48–51) and photonic materials (52, 53) and serve as a powerful tool for quantum control. Taken together, these developments call for a comprehensive theory for temporal

Significance

Just as waves scatter at a spatial interface, they also scatter at a “time boundary” when a medium’s properties change suddenly in time. By developing a time-domain scattering theory, we uncover resonant transmissions—special wave modes that cross these temporal boundaries with perfect transmission and then dynamically freeze. Notably, these perfect transmissions are governed by the system’s underlying topology. Our findings establish a general rulebook for temporal scattering, offering strategies to manipulate quantum states and devices.

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scattering at generic TBs that unveils its fundamental dynamical principles and intrinsic connections to the topology of quantum systems.

1. Scattering Matrix at a Time Boundary

We consider a dD lattice system with time-varying parameters. A generic setting of TB is specified by a piecewise time-dependent Hamiltonian

$$H(t) = \begin{cases} H^L, & t \leq 0, \\ H^{TB}(t), & 0 < t < \tau_B, \\ H^R, & t \geq \tau_B, \end{cases} \quad [1]$$

where H^L and H^R are static Hamiltonians before and after the TB, and τ_B is the duration of the TB. The special case $\tau_B = 0$ corresponds to a sudden quantum quench. Inside the TB, $H^{TB}(t)$ is otherwise arbitrary. The lattice system has $2N$ bands. The Bloch eigenstates and energy bands of $H^{L,R}$ (ordered from low to high) are denoted by $|\psi_j^{L,R}(\mathbf{k})\rangle$ and $E_j^{L,R}(\mathbf{k})$ ($j = 1, 2, \dots, 2N$), with $\mathbf{k} = (k_1, k_2, \dots, k_d)$ the crystal momentum. We set the Fermi level to zero, which can always be done for gapped topological phases. Bloch bands below (above) it, $E_j^{L,R}(\mathbf{k})$ with $j = 1, 2, \dots, N$ ($j = N + 1, \dots, 2N$), are valence (conduction) bands with negative (positive) energies. Time evolution across the TB obeys the Schrödinger equation. The general solutions before and after the TB can be written as

$$|\Psi_{\text{before}}\rangle = \sum_{j=1}^N a_j e^{iE_j^L t} |\psi_j^L\rangle + \sum_{j=N+1}^{2N} a_j e^{-iE_j^L t} |\psi_j^L\rangle,$$

$$|\Psi_{\text{after}}\rangle = \sum_{j=1}^N b_j e^{iE_j^R t} |\psi_j^R\rangle + \sum_{j=N+1}^{2N} b_j e^{-iE_j^R t} |\psi_j^R\rangle.$$

Instead of focusing on the full time evolution and nonequilibrium quantum dynamics, we treat the TB as a genuine scattering interface between two static bulk phases with well-defined asymptotic Bloch bands at $t \leq 0$ and $t \geq \tau_B$. This conceptual shift from time evolution to time scattering is crucial for the results that follow. Fig. 1 draws an analogy between 1D spatial scattering and temporal scattering. In the former, a plane wave e^{ikx} of fixed energy scatters off a potential $V(x)$, producing reflected and transmitted waves. In temporal scattering, the crystal momentum is conserved across the TB but energy is not.

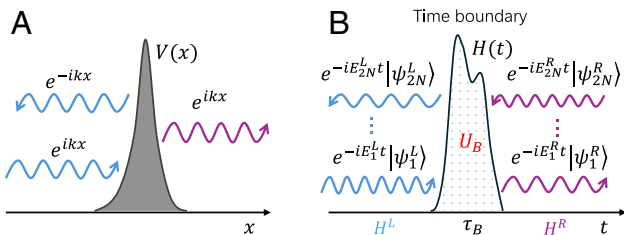


Fig. 1. Spatial versus temporal scattering. (A) In real space, a plane wave of fixed energy impinges on a potential $V(x)$, leading to reflected and transmitted waves and possibly bound states. The process is governed by stationary scattering theory. (B) In time, the Hamiltonian changes from H^L to H^R across a time boundary (TB). Bloch waves of definite crystal momentum act as scattering channels, while energy is not conserved. A TB of duration τ_B generates a unitary evolution U_B and the scattering obeys the Schrödinger equation. Arrows pointing toward (away from) the TB indicate incoming (outgoing) waves, and colors distinguish states before and after the boundary.

Excitations are produced as the system passes through the TB. The Bloch eigenstates act as $2N$ distinct scattering channels. Formally, time t plays the role of spatial coordinate x , and the plane-wave phase factor is replaced by a dynamical phase:

$$x \leftrightarrow t; \quad e^{ikx} \leftrightarrow e^{-iE_j^{L,R} t}. \quad [2]$$

During scattering, the initial wave $|\Psi_{\text{before}}\rangle$, expressed in the eigenbasis of H^L , enters the TB. Upon exit, it becomes $|\Psi_{\text{after}}\rangle$. The temporal transfer matrix M is defined by

$$(b_1, b_2, \dots, b_{2N})^T = M (a_1, a_2, \dots, a_{2N})^T. \quad [3]$$

As the scattering process follows the Schrödinger equation, the TB induces a unitary evolution $U_B = \mathbb{T} e^{-i \int_0^{\tau_B} H^{TB}(t) dt}$ ($\mathbb{T}$ means time ordering). The elements of M are

$$M_{mn} = \langle \psi_m^R | U_B | \psi_n^L \rangle. \quad [4]$$

$U_B = I$ for a sudden quench. The M -matrix, determined by H^L , H^R and the specific profile of $H^{TB}(t)$, encodes how an initial Bloch wave scatters at the boundary. Its unitarity, $M^\dagger M = M M^\dagger = I$, ensures probability conservation of the scattering process.

In scattering theory, the S -matrix is the central object that relates incoming waves on either side of a boundary to outgoing waves (including both reflected and transmitted components). To obtain it, we follow the analogy in Eq. 2, as sketched in Fig. 1. In temporal scattering, the incoming waves correspond to the valence bands of H^L (before the TB) and the conduction bands of H^R (after the TB). The outgoing waves correspond to the conduction bands of H^L and the valence bands of H^R . In spatial scattering, reflection involves momentum reversal. Here, the reflected waves refer to the conduction bands before the TB and the valence bands after the TB. This standard terminology does not imply backward time evolution, and the physical state always evolves forward in time. Furthermore, these reflected waves differ from the “reflected” components discussed in refs. 42, 45, and 41, which refer to waves that reverse their group velocity after crossing the TB (i.e., spatially propagate in the opposite direction). The S -matrix then reads

$$(b_{1 \rightarrow N}; a_{N+1 \rightarrow 2N})^T = S (a_{1 \rightarrow N}; b_{N+1 \rightarrow 2N})^T. \quad [5]$$

A straightforward calculation yields (See [SI Appendix, Appendix SI](#) for a derivation of the scattering S -matrix.)

$$S = \begin{pmatrix} M_{--} - M_{-+} M_{++}^{-1} M_{+-} & M_{-+} M_{++}^{-1} \\ -M_{++}^{-1} M_{+-} & M_{++}^{-1} \end{pmatrix}, \quad [6]$$

where M_{--} , M_{-+} , M_{+-} , and M_{++} denote the upper-left, upper-right, lower-left, and lower-right blocks of the M -matrix, respectively. The matrix elements of S encode the reflection and transmission amplitudes between different channels, and the eigenphases of S represent the associated scattering phase shifts.

2. Resonant Transmission

The S -matrix allows us to analyze the scattering process at the TB in a precise manner. In the multichannel scattering, an incident Bloch wave with crystal momentum $\mathbf{k}$ generally splits, after crossing the TB, into $2N$ outgoing branches of H^R (N in the conduction sector and N in the valence sector). In two-band systems, this manifests as birefringence. We are

particularly interested in momenta at which the TB acts as a perfect “frequency converter” between Bloch bands, converting an incoming state in one band entirely into an outgoing state in another. Within our scattering framework, these special points appear as poles of the time-domain S -matrix, which we term topological resonant transmissions (RTs). In scattering theory, poles of the S -matrix are particularly important: In stationary scattering, poles on the real-energy axis correspond to bound states. This naturally raises the question of what physical information such poles encode in the TB scattering, what nontrivial effects they generate, and how they are related to the topology. From the explicit form of the S -matrix in Eq. 6, the poles occur when M_{++} is noninvertible, i.e., $\det M_{++}(\mathbf{k}) = 0$. Under this condition, an incident wave in the valence band of H^L perfectly transitions to the conduction band of H^R across the TB. Near a pole $\mathbf{k}_0$, the norm diverges as $\|M_{++}^{-1}(\mathbf{k})\| \sim |\mathbf{k} - \mathbf{k}_0|^{-p}$, where p is the order of the pole. (See *SI Appendix, Appendix S2* for examples of higher-order poles.)

We illustrate the TB scattering and the emergence of RTs using a 2D two-band lattice model. The time-dependent Hamiltonian is

$$H(\mathbf{k}, t) = \mathbf{h}(\mathbf{k}) \cdot \boldsymbol{\sigma} + m_x(t)\sigma_x + m_z(t)\sigma_z, \quad [7]$$

where $\mathbf{h}(\mathbf{k}) = (\sin k_x, \sin k_y, -\cos k_x - \cos k_y)$ and $\boldsymbol{\sigma}$ denotes the Pauli matrices. The TB is implemented over the interval $t \in (0, \tau_B)$, with time-varying mass terms $m_x(t) = \frac{1}{2} \sin \frac{t}{\tau_B}$ and $m_z(t) = \left(1 - \frac{t}{\tau_B}\right)m^L + \frac{t}{\tau_B}m^R$, as shown in Fig. 2A. On either side of the TB, the Hamiltonian reduces to the paradigmatic Qi–Wu–Zhang model of a Chern insulator (54), with mass $m_z = m^L$ and $m_z = m^R$, respectively. The band topology is characterized by the Chern number, $C = \text{sgn}(m_z)$ for $0 < |m_z| < 2$ and $C = 0$ otherwise. In what follows, we fix $m^L = 1$ and $m^R = 3$, so that the band topology changes from $C = 1$ to $C = 0$ across the TB.

For each fixed TB duration τ_B , there is a single pole in the 2D BZ. Its trajectory as a function of τ_B is shown in Fig. 2B.

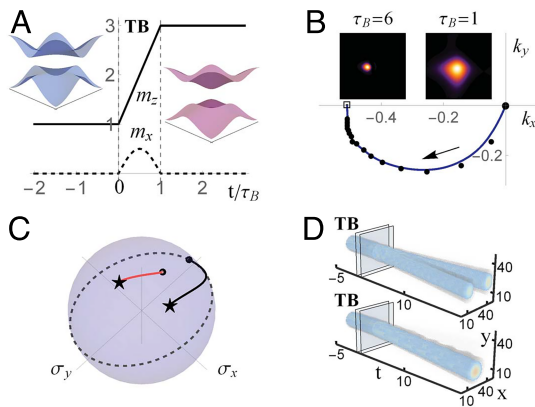


Fig. 2. Resonant transmission (RT) across a TB for the 2D model in Eq. 7. (A) Schematic of the TB with time-varying mass terms $m_x(t)$ and $m_z(t)$. Band structures on either side are shown in blue and purple. (B) Trajectory of the poles of the S -matrix in the Brillouin zone (BZ) as the TB duration τ_B increases (indicated by the arrow). Contour plots show the interchannel transmission T_{-+} for two different τ_B values. The brightest points mark the RT. (C) Time evolution on the Bloch sphere for the RT mode (red) and an off-resonant mode $\mathbf{k} = (0.5, 0)$ (black). The system starts in the ground state of H^L (black stars). The evolution inside and after the TB is shown by the solid and dashed lines, respectively, with the black dots marking the points when the state exits the TB. (D) Scattering of Gaussian wave packets. (Top) $\mathbf{k} = (0.5, 0)$; (Bottom) RT mode, $\mathbf{k} = (-0.144, -0.24)$. Parameters: packet center at $\mathbf{r}_0 = (25, 25)$, width $\sigma = 6$, and system size $L_x = L_y = 50$. $\tau_B = 1$ for (C and D).

At $\tau_B = 0$ (a quenched TB), the pole sits at the zone center $\mathbf{k} = (0, 0)$; as τ_B increases, it gradually moves in the BZ. No pole appears if H^L and H^R lie in the same topological phase. We next examine the momentum-dependent transmission across the TB. Starting from the ground state of H^L , $|\psi_-^L(\mathbf{k})\rangle$ at $t = 0$, the TB scatters it into both eigenchannels of H^R . The interchannel transmission is characterized by

$$T_{-+}(\mathbf{k}) = |\langle \psi_+^R(\mathbf{k}) | U_B | \psi_-^L(\mathbf{k}) \rangle|^2, \quad [8]$$

where U_B represents the time evolution inside the TB. Fig. 2B also shows the distribution of $T_{-+}(\mathbf{k})$ in the BZ for two representative TB durations. One clearly sees that perfect transfer occurs at RT mode. As τ_B increases, the resonant peak becomes sharper but never vanishes.

A key physical consequence of RT is the dynamical freezing of the quantum state in time evolution. Consider the time-evolved state $|\psi(t)\rangle = \mathbb{T} e^{-i \int_0^t H(t') dt'} |\psi_-^L(\mathbf{k})\rangle$. For a two-level system, this state can be represented by a single point $(\psi(t) | \boldsymbol{\sigma} | \psi(t))$ on the Bloch sphere. As shown in Fig. 2C, for the RT mode, the state moves on the Bloch sphere when it is inside the TB. Upon exit, it no longer evolves and instead becomes frozen at a fixed point on the sphere. In contrast, for a typical off-resonant mode, the evolution persists as Larmor precession after exiting the TB. These differences are also reflected in the wave-packet dynamics, as shown in Fig. 2D. We prepare a Gaussian wave packet with central momentum $\mathbf{k}$, as can be realized in cold-atom experiments (45). The initial wave function is $|\psi(\mathbf{r}, t = 0)\rangle = \eta(\mathbf{r}) \otimes |\psi_-^L(\mathbf{k})\rangle$, where

$$\eta(\mathbf{r}) \propto e^{-\frac{(\mathbf{r}-\mathbf{r}_0)^2}{2\sigma^2}} e^{i\mathbf{k} \cdot (\mathbf{r}-\mathbf{r}_0)} \quad [9]$$

describes an envelope centered at $\mathbf{r}_0 = (x_0, y_0)$ with width σ . For an off-resonant $\mathbf{k}$, the wave packet exhibits birefringence, splitting into two branches with opposite group velocities. By contrast, for the RT momentum, the packet passes through the TB transparently without birefringence.

3. Bulk-Time-Boundary Correspondence

The RT can appear in gapped topological phases in any spatial dimension. (See *SI Appendix, Appendix S3* for the case of 3D chiral topological insulators.) We focus on d D integer AZ classes, whose Hamiltonian admits an elementary representation via Clifford algebra:

$$H(\mathbf{k}) = \sum_{a=0}^d h_a(\mathbf{k}) \Gamma^a. \quad [10]$$

The AZ class is specified by three fundamental symmetries: time-reversal $\mathcal{T}$, particle-hole $\mathcal{P}$, and chiral $\mathcal{C}$. The Γ matrices reduce to Pauli matrices in the previous example. Let $\mathbf{h} = (h_0, h_1, \dots, h_d)$, then $\tilde{\mathbf{h}} = \frac{\mathbf{h}}{|\mathbf{h}|}$ defines a map from the BZ, a d D torus T^d , to the d -sphere S^d . The topological invariant n is the degree of this map, given by the winding or Chern number for odd or even d .

Remarkably, the number of RT modes is not a microscopic detail but a topological quantity. For the case of quenched TB connecting two phases with topological invariants n_L and n_R , one can show that the number of topologically protected RT modes in the d D BZ, denoted by N_{RT} , is given by the difference of topological invariants across the TB:

$$N_{RT} \equiv \sum_{\mathbf{k}_0 \in RT} p_{\mathbf{k}_0} = |n_L - n_R|. \quad [11]$$

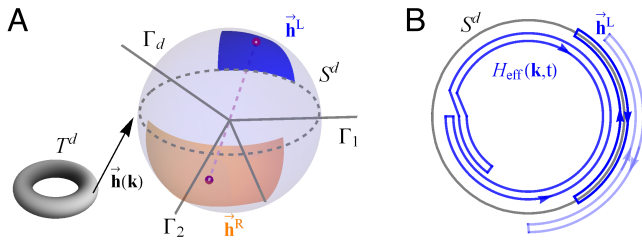


Fig. 3. Topological view of RT in the elementary Clifford representation. (A) For integer AZ classes in dD , the topological invariant is the mapping degree from the Brillouin zone (T^d) to the d sphere (S^d) spanned by the normalized Hamiltonian vector $\vec{h}$. A RT arises at a quenched TB when $\vec{h}^L$ and $\vec{h}^R$ are antiparallel. (B) Stability of RTs in even dimensions. Temporal modulations inside the TB can stretch the mapping encoded by the effective Hamiltonian $H_{\text{eff}}(\mathbf{k}, t)$, but cannot alter the global topology without gap closings.

Here $p_{\mathbf{k}_0}$ is the pole order of the protected RT at $\mathbf{k}_0$. Accidental resonances may occur even when the initial and final phases are topologically equivalent ($n_L = n_R$). However, these accidental RTs lack topological protection and can be easily destroyed by smooth perturbations without closing the bulk band gap. (See *SI Appendix, Appendix S4* for a concrete example.) A geometric view of the RT is illustrated in Fig. 3A. Consider the interpolation between the two Hamiltonians on either side of the TB: $\mathbf{h}_\lambda(\mathbf{k}) = (1 - \lambda)\mathbf{h}^L(\mathbf{k}) + \lambda\mathbf{h}^R(\mathbf{k})$, $\lambda \in [0, 1]$. If H^L and H^R differ topologically, the interpolation undergoes $|n_L - n_R|$ gap closings at the Fermi level as λ varies from 0 to 1. At each closing, $\mathbf{h}_\lambda(\mathbf{k}) = 0$. This means that the two Hamiltonian vectors $\vec{h}^L$ and $\vec{h}^R$ are antiparallel to each other on the sphere S^d . The corresponding $\mathbf{k}$ then represents an RT mode where the valence and conduction bands of H^L and H^R exchange. A proof for generic multiband systems, without relying on the elementary Clifford representation, is provided in *Materials and Methods*.

We have seen that the RTs lead to dynamical freezing in the time evolution and result in vanishing birefringence. What is the counterpart of these RTs in the spatial scattering? In stationary scattering theory, Levinson's theorem (55, 56) is a fundamental and elegant relation between the scattering phase shift and the number of bound states N_{bound} :

$$N_{\text{bound}} = \frac{1}{\pi} [\delta(0) - \delta(\infty)] = \frac{1}{\pi} \left| \int_0^\infty \frac{\partial \delta(E)}{\partial E} dE \right|, \quad [12]$$

where $\delta(E)$ is the scattering phase shift at energy E . Physically, the theorem implies that the bound state leaves a fingerprint in the continuum scattering states. For the TB, we identify a dual counting principle for RTs. From the S -matrix in Eq. 6, we have (*Materials and Methods*)

$$N_{RT} = \frac{1}{2\pi} \left| \int_{-\pi}^\pi \frac{\partial \delta_{\text{rel}}(k)}{\partial k} dk \right|, \quad [13]$$

where $\delta_{\text{rel}}(k) = \arg[\det M]$ is the relative phase shift between valence and conduction channels. We refer to this as time-domain Levinson's theorem, given the duality $E \leftrightarrow k$. As $\delta_{\text{rel}}(k)$ depends on the gauge choices of the Bloch states, we need to fix the gauge to ensure this phase winding is a well-defined physical quantity. Because the physical lattice does not shift across the TB, H^L and H^R share the same spatial coordinate system. Evaluating the initial and final Bloch states under a common gauge convention anchored to this shared spatial basis eliminates the gauge ambiguity. The theorem states that the winding of relative phase shift yields the number of RTs. Thus, the RT

mode is analogous to bound states at spatial interfaces, while off-resonant modes are akin to scattering states. Eq. 11 can be viewed as a bulk-time-boundary correspondence. The number of RT modes is determined by the jump of the bulk topological invariant across the TB, in direct analogy with domain-wall states at spatial interfaces between topologically distinct phases, where the number is given by the difference of bulk invariants on either side. Despite the duality, we note a key difference: The spatial Levinson's theorem counts bound states isolated from the continuum spectra, whereas for the TB, the RT modes are dynamically frozen and embedded in the continuum BZ.

Next, we consider the scenario of generic TB settings and discuss the robustness of RTs against temporal modulations inside the TB. A striking outcome of our analysis is that RT robustness crucially depends on spatial dimensionality. For each AZ class, we assume that the instantaneous Hamiltonian $H^{TB}(t)$ inside the TB preserves the relevant symmetries. By robustness, we mean that the number of RTs remains fixed by the topological difference between H^L and H^R , even though the corresponding momentum modes hosting RTs may shift. In such generic settings, an additional unitary transformation acts on H^L (Eq. 4). The time-evolved states $|\psi_n^L(t)\rangle$ within the TB can be regarded as eigenstates of an effective Hamiltonian

$$H_{\text{eff}}(\mathbf{k}, t) = U(0, t)H^L(\mathbf{k})U^\dagger(0, t), \quad t \in (0, \tau_B), \quad [14]$$

with $U(0, t) = \mathbb{T}e^{-i \int_0^t H(t') dt'}$. According to the bulk-time-boundary correspondence in Eq. 11, the RT is determined by the topological difference between H^R and $H_{\text{eff}}(\mathbf{k}, \tau_B)$. As the spectrum of $H_{\text{eff}}(\mathbf{k}, t)$ is fixed (and gapped) inside the TB, its topology is unchanged as long as the relevant symmetries are preserved.

For the three symmetries of all AZ classes, we have $\mathcal{T}H_{\text{eff}}(\mathbf{k}, t)\mathcal{T}^{-1} \neq H_{\text{eff}}(-\mathbf{k}, t)$; $\mathcal{P}H_{\text{eff}}(\mathbf{k}, t)\mathcal{P}^{-1} = -H_{\text{eff}}(-\mathbf{k}, t)$; $\mathcal{C}H_{\text{eff}}(\mathbf{k}, t)\mathcal{C}^{-1} \neq -H_{\text{eff}}(\mathbf{k}, t)$. (See *SI Appendix, Appendix S5* for details on AZ symmetry classes.) Namely, the $\mathcal{T}$ and $\mathcal{C}$ symmetries can be broken by the temporal modulation within the TB, while the $\mathcal{P}$ symmetry cannot. It follows that the RTs are robust in class A (no symmetry, $d = 2, 4, 6, 8$), D ($\mathcal{P}^2 = 1$, $d = 2, 6$), and C ($\mathcal{P}^2 = -1$, $d = 2, 6$). This topological robustness is intuitively visualized in Fig. 3B. Although the transformation in Eq. 14 deforms the mapping $T^d \rightarrow S^d$ of $\vec{h}^L$, the topological invariant n_L remains intact without any gap closing, due to the preservation of symmetry for these symmetry classes. In contrast, for classes AIII, BDI, DIII, CII, and CI which possess $\mathcal{C}$ symmetry, the temporal modulation inside the TB can alter the topology of $H_{\text{eff}}(\mathbf{k}, t)$, rendering the RT unstable. Such change in topology resembles a rubber band adjusting its winding along the equator as it stretches over a sphere. Note that these five classes support integer invariants only in odd d , consistent with the fact that the winding numbers depend on the transformation $U(0, t)$. For class AI and AII, the breaking of $\mathcal{T}$ symmetry reduces them to class A. Their Chern number is unaffected by the transformation $U(0, t)$ without gap closings, and the RT is stable. To sum up, the RT is robust for integer AZ classes (A, D, C, AI, AII) in even d . In contrast, for odd d (AIII, BDI, DIII, CII, CI), RT stability requires additional constraints evading dynamical breaking of symmetries.

4. Robustness Against Disorder

The emergence of RT as a topological effect is tied to the topological change across the TB and should be robust against

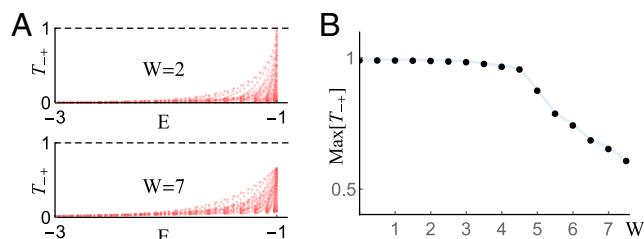


Fig. 4. Stability of RT in the presence of disorder for model [7]. (A) Interchannel transmission probability T_{-+} versus eigenenergy of H^L for two different disorder strengths, (Top) $W = 2$ and (Bottom) $W = 7$. The horizontal dashed line serves as a guide. (B) Maximum interchannel transmission probability (among all valence-band eigenstates of H^L) to the conduction bands of H^R versus the disorder strength W .

disorder. Although RT is formulated in terms of Bloch states in momentum space, its physical effects persist even in the presence of disorder, which breaks lattice translation symmetry. To study the interplay between disorder and TB effects, we introduce Anderson-type disorder into the Hamiltonian after the TB, with the disorder term $H_{\text{dis}} = \sum_j w_j c_j^\dagger c_j$, where $w_j \in [-W/2, W/2]$ and W is the disorder strength. We note that the conclusion holds for other disorder models or disorder applied over other temporal stages. The stability of RT is independent of the microscopic details of the disorder, as long as the relevant symmetries are respected.

For disordered systems, we consider eigenstates in the valence (negative-energy) bands of H^L , labeled by their eigenenergy $E = E_j^L < 0$, and compute their transmission probability $T_{-+}(E)$ to the conduction (positive-energy) bands of H^R after crossing the TB. As shown in Fig. 4A, for weak disorder $W = 2$, the transmission probability $T_{-+}(E)$ exhibits noticeable fluctuations as a function of the eigenenergy, yet a prominent transmission peak with $T_{-+}(E) \approx 1$ appears at the band edge. This suggests that there is always a certain eigenstate of H^L that, after scattering at the TB, fully transfers to the conduction band of H^R . For strong disorder $W = 7$, this peak disappears, indicating that sufficiently strong disorder destroys the RT. In Fig. 4B, we plot the maximum transmission among all negative-energy eigenstates of H^L as a function of disorder strength W . It is evident that perfect transmission survives up to a considerable disorder strength ($W \approx 4.5$) and then collapses in the strong disorder regime. These numerical results highlight the topological robustness of RT against disorder.

5. Dynamical Symmetry Breaking

In odd dimensions, the scattering process at the TB can be accompanied by dynamical symmetry breaking, which disrupts the RT. We demonstrate this instability against temporal modulations within the TB using the 1D extended Su–Schrieffer–Heeger model. The Hamiltonian is given by

$$H = \begin{pmatrix} 0 & 1 + \gamma_1 e^{-ik} + \gamma_2 e^{-2ik} \\ 1 + \gamma_1 e^{ik} + \gamma_2 e^{2ik} & 0 \end{pmatrix}. \quad [15]$$

The model consists of two sublattices per unit cell, with γ_1 (γ_2) representing nearest (next-nearest) intercell hoppings. The model belongs to the BDI class and respects chiral symmetry, $\mathcal{C} = \sigma_z$. The phase diagram in Fig. 5A shows different phases labeled by the integer winding number w_1 .

For a quenched TB ($\gamma_1^L, \gamma_2^L \rightarrow \gamma_1^R, \gamma_2^R$), poles in the S -matrix appear when the TB connects two phases with different winding

numbers. Fig. 5B shows the interchannel transmission $T_{-+}(k)$ for three different quench protocols: ① $(0.5, 0.5) \rightarrow (2.5, 0.5)$, ② $(0.5, 0.5) \rightarrow (0.5, 2)$, and ③ $(0.5, 0.5) \rightarrow (1, 0.5)$. The RT mode is located at $k = \pi$ with perfect transfer $T_{-+}(\pi) = 1$ in case ①. In case ②, there are two such RT modes. In contrast, for case ③, no pole exists, and $T_{-+}(k)$ remains below unity for all momenta. For all three cases, the number of RTs aligns with the difference of winding numbers across the TB. We then modulate the TB in case ① by assigning it a finite duration. Within the TB, two cases are studied: i) $H^{TB}(t) = \sigma_z$, which satisfies $[H^{TB}(t), \mathcal{C}] = 0$; and ii) $H^{TB}(t) = (1 - t/\tau_B)H^L + t/\tau_B H^R + \sigma_z$. As shown in Fig. 5B, in case i), the RT peak remains but shifts away from π ; in case ii), the RT vanishes due to the breaking of the chiral symmetry inside the TB.

6. Discussions

To conclude, we have developed a unified framework for temporal scattering of Bloch waves at generic TBs. By drawing a close analogy between spatial and temporal scattering, we identify topologically protected RTs that emerge when the time-varying Hamiltonian undergoes a topological transition across the TB. These RTs, corresponding to poles of the S -matrix, give perfect interchannel state transfer, vanishing birefringence, and a dynamically frozen quantum state. We establish a bulk–time–boundary correspondence, showing that the number of RTs is fixed by the jump of topological invariants across the TB, and in 1D, satisfies a time-domain Levinson’s theorem. We also reveal a striking even–odd effect in the topological stability of RTs in different spatial dimensions. Our work offers a scattering-based perspective on time-varying physics by treating the temporal region as a genuine scattering interface between two static bulk phases. This shift enables the study of nonequilibrium quantum dynamics within a standard scattering framework.

Our theory bridges temporal and spatial scattering and opens avenues for steering quantum systems through temporal engineering. It reveals how well-designed TBs can selectively enhance, suppress, or freeze specific modes, enabling precise control of quantum dynamics. While we focus on integer AZ classes, it is promising to extend the framework to more intricate topological systems, e.g., $\mathbb{Z}_2$ AZ classes (57–59), topological crystalline insulators/superconductors (60–62), and higher-order topological phases (63–65). Furthermore, the scattering dynamics and RTs serve as tantalizing tools for identifying topological

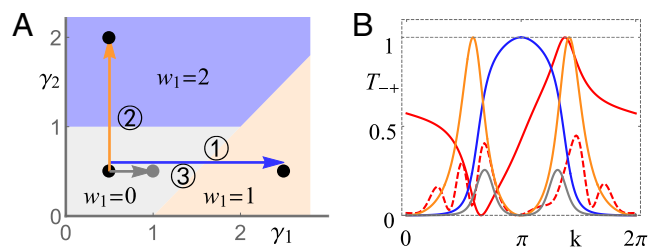


Fig. 5. RT in the extended SSH model [15] and dynamical symmetry breaking. (A) Phase diagram with topologically distinct phases (colored regions) labeled by the winding number w_1 . Arrows indicate the three quenched TBs. (B) Interchannel transmission probability T_{-+} versus momentum k for different TB settings. The blue, orange, and gray lines correspond to the quenched protocols ①, ②, and ③ in (A). The solid and dashed red lines are for temporal modulation $H^{TB}(t) = \sigma_z$ and $H^{TB}(t) = (1 - t/\tau_B)H^L + t/\tau_B H^R + \sigma_z$, respectively.

phases. For instance, in cold atoms, a TB can be placed between a known trivial system (e.g., with large detunings or deep lattices) and the target system. According to the bulk–time–boundary correspondence, RTs appear when the target system is topological. They manifest as the vanishing birefringence in the scattering pattern; and dynamical freezing can be detected via time-resolved Bloch-state tomography (66–69). Since RTs typically occur at high-symmetry points, a few momentum-resolved measurements suffice to determine the phase. This is broadly applicable and arguably one of the most convenient probes of topology, compared with: i) static approaches requiring well-prepared ground states and full BZ scans; ii) topological response relying on transport setups tailored to specific phases such as Chern insulators; or iii) dynamical schemes such as polarization measurements (20) on the $(d-1)$ D band-inversion surfaces (30).

7. Materials and Methods

7.1. Resonant Transmission in Generic Multiband Systems. In the main text, we present a heuristic topological argument for the bulk–time–boundary correspondence, which states that the number of resonant transmissions (RTs) is given by the jump in topological invariants across the time boundary (TB), i.e., Eq. 11. This argument relies on an elementary Clifford representation of the Bloch Hamiltonians. Here, we show that this identity holds in generic multiband systems, independent of any Clifford representation. Let n_L and n_R denote the topological invariants on either side of the TB, i.e., winding numbers in odd spatial dimensions and Chern numbers in even spatial dimensions. The basic setup is a generic $2N$ -band system with a quenched TB. For both H^L and H^R , a band gap separates the valence bands (labeled from 1 to N) from the conduction bands (labeled from $N+1$ to $2N$). We set the Fermi level to zero; otherwise one can shift the overall energy without changing the eigenstates, and hence without affecting the transfer matrix M or the scattering matrix S .

Let us expand the Hamiltonian in the eigenbasis as $H^{L,R} = \sum_{j=1}^{2N} E_j^{L,R} |\psi_j^{L,R}\rangle \langle \psi_j^{L,R}|$. The spectra can be “flattened” without closing the gap or altering the eigenstates $\tilde{H}^{L,R} = \sum_{j=1}^N (-1) |\psi_j^{L,R}\rangle \langle \psi_j^{L,R}| + \sum_{j=N+1}^{2N} (+1) |\psi_j^{L,R}\rangle \langle \psi_j^{L,R}|$. In terms of the projection operators onto the conduction and valence subspaces: $P_+^{L,R} = \sum_{j=N+1}^{2N} |\psi_j^{L,R}\rangle \langle \psi_j^{L,R}|$, $P_-^{L,R} = 1 - P_+^{L,R}$, the flattened Hamiltonian is

$$\tilde{H}^{L,R} = P_+^{L,R} - P_-^{L,R} = 2P_+^{L,R} - 1. \quad [16]$$

The above procedure leaves M and S unchanged and preserves the topology. Consider an interpolation between the two flattened Hamiltonians, $\tilde{H}_\lambda = (1-\lambda)\tilde{H}^L + \lambda\tilde{H}^R$, $\lambda \in [0, 1]$. As λ varies from 0 to 1, the topological invariant changes from n_L to n_R (by assumption $n_L \neq n_R$). As the topology can only change when the energy gap between the conduction and valence bands closes, there must be $|n_L - n_R|$ nontrivial gap closings along the interpolation. If a band closing involves higher-order degeneracies (e.g., quadratic), its multiplicity should be counted.

Suppose a gap closing occurs at $\lambda = \lambda_0 \in (0, 1)$ and $\mathbf{k} = \mathbf{k}_0$. At this point, there exists a nonzero state $|\phi\rangle$ such that

$$\tilde{H}_{\lambda_0}(\mathbf{k}_0)|\phi\rangle = 0. \quad [17]$$

The choice of $|\phi\rangle$ is not unique, and the degree of freedom comes from the superposition in the degenerate subspace. Substituting Eq. 16 into Eq. 17, we have

$$(1-\lambda_0)P_+^L|\phi\rangle + \lambda_0 P_+^R|\phi\rangle = \frac{1}{2}|\phi\rangle. \quad [18]$$

Letting $|x\rangle = P_+^L|\phi\rangle$ and $|y\rangle = P_+^R|\phi\rangle$, and applying P_+^R to the left side, Eq. 18 becomes

$$(1-\lambda_0)P_+^R|x\rangle + (\lambda_0 - \frac{1}{2})|y\rangle = 0. \quad [19]$$

Next, we show that the gap closing necessarily implies $\det M_{++} = 0$, i.e., a pole of the S -matrix. Here M_{++} is the conduction–conduction block of the transfer matrix. Let us expand $|x\rangle$ in the basis of left conduction states, $|x\rangle = \sum_{j=1}^N c_j |\psi_{N+j}^L\rangle$. Then in the basis of the right conduction states, we have $\langle \psi_{N+i}^R | P_+^R | x \rangle = \sum_{j=1}^N c_j \langle \psi_{N+i}^R | \psi_{N+j}^L \rangle = \sum_{j=1}^N (M_{++})_{ij} c_j$.

We discuss two cases of λ_0 in Eq. 19. If $\lambda_0 = 1/2$, we have $P_+^R|x\rangle = 0$, which immediately implies that $\det M_{++} = 0$. If $\lambda_0 \neq 1/2$, then $|y\rangle = \frac{\lambda_0-1}{\lambda_0-1/2} P_+^R|x\rangle$. In this case, we show that M_{++} is noninvertible by contradiction. If M_{++} were invertible, $|x\rangle$ and $|y\rangle$ would be linearly independent. Applying P_+^L to the left side of Eq. 17 yields $(\frac{1}{2} - \lambda_0)|x\rangle + \lambda_0 P_+^L|y\rangle = 0$. Noting that $P_+^R P_+^L = M_{++}^\dagger M_{++}$ on the left conduction bands, we have $|x\rangle = \frac{\lambda_0(\lambda_0-1)}{(\lambda_0-1/2)^2} P_+^R P_+^L|x\rangle = \frac{\lambda_0(\lambda_0-1)}{(\lambda_0-1/2)^2} (M_{++}^\dagger M_{++})|x\rangle$. Since $M_{++}^\dagger M_{++}$ is positive definite and $0 < \lambda_0 < 1$, the only possibility is the trivial solution $|x\rangle = 0$, which forces $|y\rangle = 0$ and $\phi = 0$ according to Eq. 18. Therefore, for a nontrivial gap closing, M_{++} must be singular and $\det M_{++} = 0$. Near $\mathbf{k}_0$, the block $M_{++}(\mathbf{k})$ is a smooth matrix-valued function of $\mathbf{k}$ whose rank drops at $\mathbf{k}_0$. If the band closing has order m in the sense that the low-energy Hamiltonian first develops nonzero terms at order $(\delta k)^m$, the smallest nonzero singular value of $M_{++}(\mathbf{k})$ scales as $(\delta k)^m$. This is nothing but the pole order $\rho_{\mathbf{k}_0}$ of the scattering matrix at $\mathbf{k}_0$. Summing up, we have proved the identity Eq. 11.

7.2. Proof of Time-Domain Levinson's Theorem. The time-domain Levinson's theorem (Eq. 13) relates the scattering phase shifts to the number of topologically protected RTs. The phase shifts are given by the diagonal elements of the S -matrix in Eq. 6. For block matrices, we have $\det M = \det M_{++} \det(M_{--} - M_{-+} M_{++}^{-1} M_{+-})$. The relative phase shift between the valence and conduction bands is therefore $\delta_{rel}(k) = \arg \det(M_{--} - M_{-+} M_{++}^{-1} M_{+-}) - \arg \det(M_{++}^{-1}) = \arg \det M(k)$. Let $U_L(k)$ and $U_R(k)$ be the unitary matrices whose columns are the eigenstates of $H^L(k)$ and $H^R(k)$, ordered from valence to conduction bands. The transfer matrix is then $M(k) = U_R^\dagger(k) U_L(k)$. It is unitary, $M^\dagger M = M M^\dagger = I$, $\det M(k) = \det U_R^\dagger(k) \det U_L(k) = e^{-i\phi^R(k)} e^{i\phi^L(k)}$, where $\phi^{L,R}(k) = \arg \det U_{L,R}(k)$. The relative phase shift is $\delta_{rel}(k) = \phi^L(k) - \phi^R(k)$.

Since $\phi^L(k)$ and $\phi^R(k)$ depend on the gauge choices of the Bloch states on the two sides of the TB, a specific gauge must be chosen to ensure the integral becomes a well-defined physical quantity. Considering that the physical lattice does not shift across the TB, H^L and H^R share the same spatial coordinate system. Evaluating both $U_L(k)$ and $U_R(k)$ under a common gauge convention anchored to this shared spatial basis eliminates the gauge ambiguity. For instance, one can explicitly fix the gauge by requiring a specific component of the Bloch wave functions to be real and positive for both H^L and H^R . With this gauge fixing condition, the integral of the relative phase shift yields the difference of the winding numbers of the two phases: $\frac{1}{2\pi} \int_{-\pi}^{\pi} \partial_k \delta_{rel}(k) dk = \frac{1}{2\pi} \int_{-\pi}^{\pi} \partial_k \phi^L(k) dk - \frac{1}{2\pi} \int_{-\pi}^{\pi} \partial_k \phi^R(k) dk = n_L - n_R$, where n_L and n_R are the respective winding numbers at the two sides of the TB. Combining this with Eq. 11, we arrive at the time-domain Levinson's theorem.

Data, Materials, and Software Availability. All data and numerical codes supporting the findings of this study are publicly available in Zenodo (70).

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1. S. A. Akhmanov, A. P. Sukhorukov, A. S. Chirkin, Nonstationary phenomena and space-time analogy in nonlinear optics. *Sov. Phys. JETP* **28**, 748 (1969).
2. B. H. Kolner, Space-time duality and the theory of temporal imaging. *IEEE J. Quantum Electron.* **30**, 1951 (1994).
3. J. T. Mendonça, P. K. Shukla, Time refraction and time reflection: Two basic concepts. *Phys. Scr.* **65**, 160 (2002).
4. Y. Z. Xiao, D. N. Maywar, G. P. Agrawal, Reflection and transmission of electromagnetic waves at a temporal boundary. *Opt. Lett.* **39**, 574 (2014).
5. B. W. Plansinis, W. R. Donaldson, G. P. Agrawal, What is the temporal analog of reflection and refraction of optical beams? *Phys. Rev. Lett.* **115**, 183901 (2015).
6. S. Vezzoli *et al.*, Optical time reversal from time-dependent epsilon-near-zero media. *Phys. Rev. Lett.* **120**, 043902 (2018).
7. X. Wang *et al.*, Metasurface-based realization of photonic time crystals. *Sci. Adv.* **9**, eadg7541 (2023).
8. T. R. Jones, A. V. Kildishev, M. Segev, D. Peroulis, Time-reflection of microwaves by a fast optically controlled time-boundary. *Nat. Commun.* **15**, 6786 (2024).
9. H. Moussa *et al.*, Observation of temporal reflection and broadband frequency translation at photonic time interfaces. *Nat. Phys.* **19**, 863 (2023).
10. Y. Zhou *et al.*, Broadband frequency translation through time refraction in an epsilon-near-zero material. *Nat. Commun.* **11**, 2180 (2020).
11. J. Bohn, T. S. Luk, S. Horsley, E. Hendry, Spatiotemporal refraction of light in an epsilon-near-zero indium tin oxide layer: Frequency shifting effects arising from interfaces. *Optica* **8**, 1532 (2021).
12. B. L. Kim, C. Chong, C. Daraio, Temporal refraction in an acoustic phononic lattice. *Phys. Rev. Lett.* **133**, 077201 (2024).
13. V. Bacot, M. Labousse, A. Eddi, M. Fink, E. Fort, Time reversal and holography with spacetime transformations. *Nat. Phys.* **12**, 972 (2016).
14. H. Li, S. Yin, A. Alù, Nonreciprocity and faraday rotation at time interfaces. *Phys. Rev. Lett.* **128**, 173901 (2022).
15. Z. Hayran, J. B. Khurgin, F. Monticone, $\hbar\omega$ versus $\hbar k$: Dispersion and energy constraints on time-varying photonic materials and time crystals. *Opt. Mater. Express* **12**, 3904 (2022).
16. A. Boltasseva, V. M. Shalae, M. Segev, Photonic time crystals: From fundamental insights to novel applications: Opinion. *Opt. Mater. Express* **14**, 592 (2024).
17. M. M. Asgari *et al.*, Theory and applications of photonic time crystals: A tutorial. *Adv. Opt. Photonics* **16**, 958 (2024).
18. A. Dikopoltsev *et al.*, Light emission by free electrons in photonic time-crystals. *Proc. Natl. Acad. Sci. U.S.A.* **119**, e2119705119 (2022).
19. Y. Ren *et al.*, Observation of momentum-gap topology of light at temporal interfaces in a time-synthetic lattice. *Nat. Commun.* **16**, 707 (2025).
20. W. Sun *et al.*, Uncover topology by quantum quench dynamics. *Phys. Rev. Lett.* **121**, 250403 (2018).
21. N. Fläschner *et al.*, Observation of dynamical vortices after quenches in a system with topology. *Nat. Phys.* **14**, 265 (2018).
22. M. Tarnowski *et al.*, Measuring topology by dynamics: Chern number from linking number. *Nat. Commun.* **10**, 1728 (2019).
23. K. Wang *et al.*, Observation of emergent momentum-time skyrmions in parity-time-symmetric non-unitary quench dynamics. *Nat. Commun.* **10**, 2293 (2019).
24. K. Wang *et al.*, Simulating dynamic quantum phase transitions in photonic quantum walks. *Phys. Rev. Lett.* **122**, 020501 (2019).
25. M. D. Caio, N. R. Cooper, M. J. Bhaseen, Quantum quenches in chern insulators. *Phys. Rev. Lett.* **115**, 236403 (2015).
26. Y. Hu, P. Zoller, J. C. Budich, Dynamical buildup of a quantized hall response from nontopological states. *Phys. Rev. Lett.* **117**, 126803 (2016).
27. J. H. Wilson, J. C. W. Song, G. Refael, Remnant geometric hall response in a quantum quench. *Phys. Rev. Lett.* **117**, 235302 (2016).
28. C. Wang, P. Zhang, X. Chen, J. Yu, H. Zhai, Scheme to measure the topological number of a chern insulator from quench dynamics. *Phys. Rev. Lett.* **118**, 185701 (2017).
29. J. Yu, Phase vortices of the quenched Haldane model. *Phys. Rev. A* **96**, 023601 (2017).
30. L. Zhang, L. Zhang, S. Niu, X.-J. Liu, Dynamical classification of topological quantum phases. *Sci. Bull.* **63**, 1385 (2018).
31. C. Yang, L. Li, S. Chen, Dynamical topological invariant after a quantum quench. *Phys. Rev. B* **97**, 060304(R) (2018).
32. Z. Gong, M. Ueda, Topological entanglement-spectrum crossing in quench dynamics. *Phys. Rev. Lett.* **121**, 250601 (2018).
33. M. Ezawa, Topological quantum quench dynamics carrying arbitrary Hopf and second Chern numbers. *Phys. Rev. B* **98**, 205406 (2018).
34. P.-Y. Chang, Topology and entanglement in quench dynamics. *Phys. Rev. B* **97**, 224304 (2018).
35. L. Zhang, L. Zhang, X.-J. Liu, Characterizing topological phases by quantum quenches: A general theory. *Phys. Rev. A* **100**, 063624 (2019).
36. M. McGinley, N. R. Cooper, Topology of one-dimensional quantum systems out of equilibrium. *Phys. Rev. Lett.* **121**, 090401 (2018).
37. M. McGinley, N. R. Cooper, Classification of topological insulators and superconductors out of equilibrium. *Phys. Rev. B* **99**, 075148 (2019).
38. H. Hu, E. Zhao, Topological invariants for quantum quench dynamics from unitary evolution. *Phys. Rev. Lett.* **124**, 160402 (2020).
39. H. Hu, C. Yang, E. Zhao, Quench dynamics of Hopf insulators. *Phys. Rev. B* **101**, 155131 (2020).
40. L. Zhang, W. Jia, X.-J. Liu, Universal topological quench dynamics for $\mathbb{Z}_2$ topological phases. *Sci. Bull.* **67**, 1236 (2022).
41. H. Xu, Z. Dong, L. Yuan, L. Jin, Probing bulk band topology from time boundary effect in synthetic dimension. *Phys. Rev. Lett.* **134**, 163801 (2025).
42. H. C. Wu, H. S. Xu, L. C. Xie, L. Jin, Edge state, band topology, and time boundary effect in the fine-grained categorization of chern insulators. *Phys. Rev. Lett.* **132**, 083801 (2024).
43. M. Z. Hasan, C. L. Kane, Colloquium: Topological insulators. *Rev. Mod. Phys.* **82**, 3045 (2010).
44. X.-L. Qi, S.-C. Zhang, Topological insulators and superconductors. *Rev. Mod. Phys.* **83**, 1057 (2011).
45. Z. Dong *et al.*, Quantum time reflection and refraction of ultracold atoms. *Nat. Photonics* **18**, 68 (2024).
46. T. W. B. Kibble, Topology of cosmic domains and strings. *J. Phys. A Math. Gen.* **9**, 1387 (1976).
47. W. H. Zurek, Cosmological experiments in superfluid helium? *Nature* **317**, 505 (1985).
48. C. Weitenberg, J. Simonet, Tailoring quantum gases by Floquet engineering. *Nat. Phys.* **17**, 1342 (2021).
49. A. Eckardt, Colloquium: Atomic quantum gases in periodically driven optical lattices. *Rev. Mod. Phys.* **89**, 011004 (2017).
50. N. R. Cooper, J. Dalibard, I. B. Spielman, Topological bands for ultracold atoms. *Rev. Mod. Phys.* **91**, 015005 (2019).
51. D. W. Zhang, Y. Q. Zhu, Y. X. Zhao, H. Yan, S. L. Zhu, Topological quantum matter with cold atoms. *Adv. Phys.* **67**, 253 (2018).
52. L. Lu, J. D. Joannopoulos, M. Soljačić, Topological photonics. *Nat. Photonics* **8**, 821 (2014).
53. T. Ozawa *et al.*, Topological photonics. *Rev. Mod. Phys.* **91**, 015006 (2019).
54. X.-L. Qi, Y.-S. Wu, S.-C. Zhang, Topological quantization of the spin Hall effect in two-dimensional paramagnetic semiconductors. *Phys. Rev. B* **74**, 085308 (2006).
55. N. Levinson, On the uniqueness of the potential in a Schrödinger equation for a given asymptotic phase. *Danske Vid. Selsk. Mat. Fys. Medd.* **25**, 1–29 (1949).
56. J. R. Taylor, *Scattering Theory: The Quantum Theory of Nonrelativistic Collisions* (Dover Publications, New York, 2006), pp. 226–228.
57. A. Altland, M. R. Zirnbauer, Nonstandard symmetry classes in mesoscopic normal-superconducting hybrid structures. *Phys. Rev. B* **55**, 1142 (1997).
58. A. P. Schnyder, S. Ryu, A. Furusaki, A. W. W. Ludwig, Classification of topological insulators and superconductors in three spatial dimensions. *Phys. Rev. B* **78**, 195125 (2008).
59. A. Kitaev, Periodic table for topological insulators and superconductors. *AIP Conf. Proc.* **1134**, 22 (2009).
60. C.-K. Chiu, J. C. Y. Teo, A. P. Schnyder, S. Ryu, Classification of topological quantum matter with symmetries. *Rev. Mod. Phys.* **88**, 035005 (2016).
61. L. Fu, Topological crystalline insulators. *Phys. Rev. Lett.* **106**, 106802 (2011).
62. Y. Ando, L. Fu, Topological crystalline insulators and topological superconductors: From concepts to materials. *Annu. Rev. Condens. Matter Phys.* **6**, 361 (2015).
63. W. A. Benalcazar, B. A. Bernevig, T. L. Hughes, Quantized electric multipole insulators. *Science* **357**, 61 (2017).
64. F. Schindler *et al.*, Higher-order topological insulators. *Sci. Adv.* **4**, eaat0346 (2018).
65. J. Langbehn, Y. Peng, L. Trifunovic, F. von Oppen, P. W. Brouwer, Reflection-symmetric second-order topological insulators and superconductors. *Phys. Rev. Lett.* **119**, 246401 (2017).
66. P. Hauke, M. Lewenstein, A. Eckardt, Tomography of band insulators from quench dynamics. *Phys. Rev. Lett.* **113**, 045303 (2014).
67. N. Fläschner *et al.*, Experimental reconstruction of the Berry curvature in a Floquet Bloch band. *Science* **352**, 1091 (2016).
68. T. Li *et al.*, Bloch state tomography using Wilson lines. *Science* **352**, 1094 (2016).
69. E. Alba, X. Fernandez-Gonzalvo, J. Mur-Petit, J. K. Pachos, J. J. Garcia-Ripoll, Seeing topological order in time-of-flight measurements. *Phys. Rev. Lett.* **107**, 235301 (2011).
70. H. Hu, Data and numerical codes for "Time-boundary scattering and topological resonant transmissions". Zenodo. <https://doi.org/10.5281/zenodo.21291941>. Deposited 3 September 2026.